The next best thing to being clever is being able to quote someone who is.
(Mary Pettibone Poole)

Appendix B

Efficient Computation of Homographies From Four Correspondences

The usual way to compute the parameters of a projective transform from four point coordinates is to use the inhomogeneous formulation of the projective transform (Eq. (2.11)). Using four point-correspondences $\mathbf{p}_i \leftrightarrow \hat{\mathbf{p}}_i$, we can set up an equation system (Eq. (3.2)) to solve for the homography matrix $\mathbf{H}$. However, this requires the solution of an 8×8 equation system.

Efficient algorithm

An algorithm to obtain these parameters requiring only the inversion of a 3×3 equation system is as follows. From the four point-correspondences $\mathbf{p}_i \leftrightarrow \hat{\mathbf{p}}_i$ with $(i \in \{1, 2, 3, 4\})$, compute $\mathbf{h}_1 = (\mathbf{p}_1 \times \mathbf{p}_2) \times (\mathbf{p}_3 \times \mathbf{p}_4)$, $\mathbf{h}_2 = (\mathbf{p}_1 \times \mathbf{p}_3) \times (\mathbf{p}_2 \times \mathbf{p}_4)$, $\mathbf{h}_3 = (\mathbf{p}_1 \times \mathbf{p}_4) \times (\mathbf{p}_2 \times \mathbf{p}_3)$. Also compute $\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{\mathbf{h}}_3$ using the same principle from the points $\hat{\mathbf{p}}_i$. Now, the homography matrix $\mathbf{H}$ can be obtained easily from

$$\mathbf{H} \cdot \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{h}}_1 & \hat{\mathbf{h}}_2 & \hat{\mathbf{h}}_3 \end{bmatrix}, \tag{B.1}$$

which only requires the inversion of the matrix $\begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 \end{bmatrix}$.

Proof

The validity of the efficient algorithm can be proven as follows.
First, it can be shown easily that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w}^\top \mathbf{u})\mathbf{v} - (\mathbf{v}^\top \mathbf{u})\mathbf{w}. \tag{B.2}$$

Furthermore, it is known that for the triple scalar product $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u}^\top(\mathbf{v} \times \mathbf{w})$ it holds that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$. Because $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$, it also holds that

$$[\mathbf{Hu}, \mathbf{Hv}, \mathbf{Hw}] = \det(\mathbf{H}) \cdot [\mathbf{u}, \mathbf{v}, \mathbf{w}]. \tag{B.3}$$

If we now compare

$$\begin{aligned} \mathbf{H}((\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})) &= \mathbf{H}(\mathbf{d}^\top(\mathbf{a} \times \mathbf{b})\mathbf{c} - \mathbf{c}^\top(\mathbf{a} \times \mathbf{b})\mathbf{d}) \\ &= \mathbf{H}([\mathbf{d}, \mathbf{a}, \mathbf{b}]\mathbf{c} - [\mathbf{c}, \mathbf{a}, \mathbf{b}]\mathbf{d}) \\ &= \det(\mathbf{d}, \mathbf{a}, \mathbf{b})\mathbf{Hc} - \det(\mathbf{c}, \mathbf{a}, \mathbf{b})\mathbf{Hd} \end{aligned} \tag{B.4}$$

with

$$\begin{aligned} (\mathbf{Ha} \times \mathbf{Hb}) \times (\mathbf{Hc} \times \mathbf{Hd}) &= \mathbf{Hd}^\top(\mathbf{Ha} \times \mathbf{Hb})\mathbf{Hc} - \mathbf{Hc}^\top(\mathbf{Ha} \times \mathbf{Hb})\mathbf{Hd} \\ &= [\mathbf{Hd}, \mathbf{Ha}, \mathbf{Hb}]\mathbf{Hc} - [\mathbf{Hc}, \mathbf{Ha}, \mathbf{Hb}]\mathbf{Hd} \\ &= \det(\mathbf{H})\big(\det(\mathbf{d}, \mathbf{a}, \mathbf{b})\mathbf{Hc} - \det(\mathbf{c}, \mathbf{a}, \mathbf{b})\mathbf{Hd}\big), \end{aligned} \tag{B.5}$$

we see that

$$\mathbf{H}\big((\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})\big) = \frac{1}{\det(\mathbf{H})}\big((\mathbf{Ha} \times \mathbf{Hb}) \times (\mathbf{Hc} \times \mathbf{Hd})\big). \tag{B.6}$$

Considering again the problem to compute the parameters of the homography transform, we can state the problem as finding the matrix $\mathbf{H}$ such that

$$\mathbf{Hp}_1 = c_1\hat{\mathbf{p}}_1, \quad \mathbf{Hp}_2 = c_2\hat{\mathbf{p}}_2, \quad \mathbf{Hp}_3 = c_3\hat{\mathbf{p}}_3, \quad \mathbf{Hp}_4 = c_4\hat{\mathbf{p}}_4, \tag{B.7}$$

where c_i are unknown constants that provide suitable scaling for the homogeneous coordinates. These four matrix equations have $9 + 4 = 13$ unknowns, but since they give only $3 \times 4 = 12$ constraints, any scaled version of $\mathbf{H}$ is a solution.

Since the actual value of the constants c_i do not matter, the trick is to reduce the equations such that the c_i are removed to only a single scaling

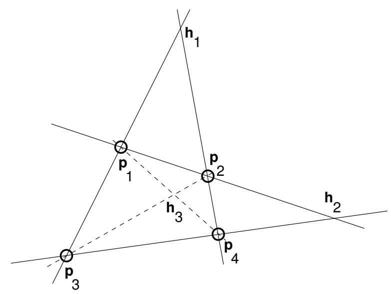

Figure B.1: *Location of the three points* $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$.

constant λ that we can choose arbitrarily. This can be achieved by considering the product $(\mathbf{p}_1 \times \mathbf{p}_2) \times (\mathbf{p}_3 \times \mathbf{p}_4)$ and all permutations up to a sign change, which gives us the vectors $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ which were defined previously. But because of Eq. (B.6),

$$\begin{aligned}
\mathbf{H}\mathbf{h}_1 &= \mathbf{H}\big((\mathbf{p}_1 \times \mathbf{p}_2) \times (\mathbf{p}_3 \times \mathbf{p}_4)\big) \\
&= \frac{1}{\det(\mathbf{H})}(\mathbf{Ha} \times \mathbf{Hb}) \times (\mathbf{Hc} \times \mathbf{Hd}) \\
&= \frac{1}{\det(\mathbf{H})}\big((c_1\hat{\mathbf{p}}_1 \times c_2\hat{\mathbf{p}}_2) \times (c_3\hat{\mathbf{p}}_3 \times c_4\hat{\mathbf{p}}_4)\big) \\
&= \underbrace{\frac{c_1 c_2 c_3 c_4}{\det(\mathbf{H})}}_{\lambda} \cdot \big((\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2) \times (\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2)\big) \\
&= \lambda \cdot \hat{\mathbf{h}}_1,
\end{aligned} \tag{B.8}$$

where we can set $\lambda = 1$, since any scaled version of $\mathbf{H}$ is a valid solution. Similar, we get $\mathbf{H}\mathbf{h}_2 = \hat{\mathbf{h}}_2, \mathbf{H}\mathbf{h}_3 = \hat{\mathbf{h}}_3$. These three equations can be combined into one equation system, giving Eq. (B.1).

It is interesting to note that the three points correspond to the intersection points of the opposite sides and the diagonals of a quadrilateral, made from the four points $\mathbf{p}_i$ (remember that the cross product of two points defines a line, and the cross product of two lines defines a point). This is visualized in Figure B.1. Clearly, the inverse is not possible, i.e., the homography cannot be determined unambiguously from just these three points.

Acknowledgement

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